B.Sc. Semester III (Honours) Examination, 2018-19

MATHEMATICS

Course ID : 32112 **Course Code : SHMTH-302C-6(T)**

Course Title : Group Theory I

Time: 2 Hours **Full Marks: 40**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

1. Answer *any five* questions: 2×5=10

(a) Let $G = \left\{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0\right\}$. Does G form a group with respect to usual matrix multiplication? Justify your answer.

(b) Prove that the subgroup $SL_n(\mathbb{R})$ is normal in the general linear group $GL_n(\mathbb{R})$.

(c) Find the subgroup generated by $\{6, 8\}$ in the group $(\mathbb{Z}, +)$.

(d) Suppose G is a cyclic group such that G has exactly three subgroups viz. G, $\{e\}$ and a subgroup of order 7. What is the order of G?

(e) Find the order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 3 & 2 & 6 \end{pmatrix}$.

(f) Determine the number of elements of order 5 in $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$.

(g) Let G be a group and $a \in G$. Prove that the mapping $f_a : G \to G$, defined by $f_a(x) = ax$ for all $x \in G$, is one to one and onto.

(h) Let G and H be two groups. Define a function $f : G \times H \to G$ by $f\big((a, b)\big) = a$ for all $(a, b) \in G \times H$. Prove that f is a homomorphism. Evaluate $\ker f$.

2. Answer *any four* questions: 5×4=20

(a) Let H be a subset of a group G and let the set $N(H)$, called the normalizer of H in G, be defined by $N(H) = \{a \in G \mid aHa^{-1} = H\}$. Prove that $N(H)$ is a subgroup of G. If in addition H be a subgroup of G, then prove that H is normal in G if and only if $N(H) = G$. 2+3=5

(b) (i) Suppose H is a finite subgroup of a group G with $O(H) = n$. If there is no other subgroup of G with n elements then prove that H is a normal subgroup of G.

(ii) Prove that alternating group A_n is a normal subgroup of the permutation group S_n. 3+2=5

(c) Let, H and K be subgroups of a group G with K normal in G. Then prove that $H/H \cap K \simeq HK/K$. 5

(d) (i) Prove that $(\mathbb{Q}, +)$ is not cyclic.

(ii) Show that every subgroup of a cyclic group is cyclic. 2+3=5

(e) (i) Let G be a group of finite order n and $a \in G$. Then prove that $\mathrm{o}(a)$ divides n and $a^n = e$.

(ii) Let p be a prime integer and a be an integer such that p does not divide a. Then using Lagrange's theorem show that $a^{p-1} \equiv 1(mod\ p)$. 2+3=5

(f) State and prove Cayley's theorem. 1+4=5

3. Answer *any one* question: 10×1=10

(a) (i) Show that any non-identity permutation $\alpha \in s_n (n \geq 2)$ can be expressed as a product of disjoint cycles, each of length ≥ 2.

(ii) Give an example (with reason) of a non-cyclic, commutative group of which each subgroup is cyclic.

(iii) If the order of a cyclic group G is divisible by a positive integer m then prove that there exists a unique subgroup of order m of G. 5+2+3=10

(b) (i) State and prove Lagrange's theorem on order of subgroup.

(ii) Show that $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic but of same cardinality.

(iii) Find all the group homomorphisms from $(\mathbb{Z}_6, +)$ into $(\mathbb{Z}_4, +)$. 4+3+3=10

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